Deptt- MATHEMATICS

College- SOGHRA COLLEGE,BIHAR SHARIF

Part- BSc PART 2

The solution is, x(I.F.) = $\int(Q\times I.F.)+C$; xlogy = $\int\frac{1}{y}(\log y)dy + c$ = $\frac{1}{2}(\log y)^2+C$

x = $\frac{1}{2}\log y + C\frac{1}{\log y}$

Illustration 33: Solve $(x + 2y^3)\frac{dx}{dy} = y$

Sol: By reducing given equation in the form of $\frac{dx}{dy}$ + Px = Q and then using the integration factor we can solve this.

$(x + 2y^3)\frac{dx}{dy} = y \Rightarrow \frac{dx}{dy} = \frac{x+2y^3}{y} = \frac{x}{y}+2y^2 \Rightarrow \frac{dx}{dy}-\frac{1}{y}x = 2y^2$

I.F = $e^{-\int\frac{1}{y}dy} = \frac{1}{y}$;

Solutions is x. $\frac{1}{y} = y^2 + C$

Alternate method: $xdy + 2y^3dy = ydx$

$\Rightarrow \quad 2ydy = \frac{ydx - xdy}{y^2} \quad \Rightarrow \quad 2ydy = d\left(\frac{x}{y}\right) \Rightarrow y^2 = \frac{x}{y} + C$

Illustration 34: Let g(x) be a differential function for every real x and g′(0) = 2 and satisfying $g(x+y) = e^y g(x) + 2e^x g(y)$ ∀ x and y. Find g(x) and its range.

Sol: By using $g'(x) = \lim_{b\to 0}\frac{g(x+h)-g(x)}{h}$ and solving we will get g(x).

$g'(x) = \lim_{b\to 0}\frac{g(x+h)-g(x)}{h}$

$\Rightarrow \quad g'(x) = \lim_{h\to 0}\frac{e^h g(x)+2e^x g(h)-g(x)}{h} \quad \Rightarrow \quad g'(x) = g(x)\lim_{h\to 0}\frac{e^h-1}{h}+2e^x\lim_{h\to 0}\frac{g(h)}{h} \quad \Rightarrow \quad g'(x) = g(x) + 2e^x$

At x = 0, g(x) = 0 ⇒ g(0) = 0

$\frac{dy}{dx} - y = 2e^x \quad \Rightarrow \quad I.F. = e^{-x}$

Solution is $y.e^{-x} = 2x + C$

$g(0) = 0 \Rightarrow C = 0 \Rightarrow g(x) = 2xe^x$

$g'(x) = 2e^x + 2xe^x = 2e^x(x + 1)$

g′(x) = 0 at x = -1; g(−1) = −2/e

$\Rightarrow \quad$ Range of g(x) = $\left[-\frac{2}{e},\infty\right]$

Illustration 35: Find the solution of $(1 - x^2)\frac{dy}{dx}+2xy = x\sqrt{1-x^2}$

Sol: By reducing given equation in the form of $\frac{dy}{dx}$ + Py = Q and then by using integration factor i.e. $e^{\int Pdx}\left(\frac{dy}{dx}+Py\right) = Qe^{\int pdx}$ we can solve the problem.

$$\frac{dy}{dx}+\frac{2x}{(1-x^2)}y=\frac{x\sqrt{1-x^2}}{1-x^2}\ ;\ \text{I.F.} = e^{\int Pdx}=e^{\int\frac{2x}{1-x^2}dx}=\frac{1}{1-x^2}$$

Solution is y. $y\frac{1}{1-x^2}=\int\frac{x}{\sqrt{1-x^2}}\frac{1}{1-x^2}dx+c\ =\int\frac{x}{(1-x^2)^{3/2}}dx+C=\frac{-1}{2}\int\frac{-2x}{\left(1-x^2\right)^{3/2}}dx+c$

$$y\frac{1}{\left(1-x^2\right)}=\frac{1}{\sqrt{1-x^2}}+c$$

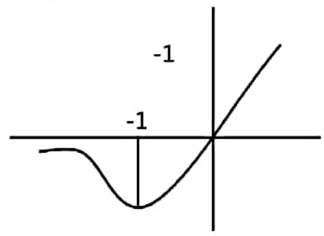

Figure 24.3

7.8 Equations Reducible to Linear form

(a) Bernoulli's Equation

A differential equation of the form $\frac{dy}{dx}$ + Py = Qy^n, where P and Q are function of x and y is called Bernoulli's equation. This form can be reduced to linear form by dividing y^n and then substituting y^{1-n} = v

Dividing both sides by y^n, we get, $y^{-n}\frac{dy}{dx}$ + $P.y^{-n+1}$ = Q

Putting y^{-n+1} = v, so that , $(1-n)y^{-n}\frac{dy}{dx}=\frac{dv}{dx}$, we get $\frac{dv}{dx}$ + (1 – n)P.y = (1 – n)Q

Which is a linear differential equation

(b) If the given equation is of the form $\frac{dy}{dx}$ + P. f(y) = Q.g(y), where P and Q are functions of x alone, we divide the equation by f(y), and then we get $e^{\int Pdx}=e^{-\ln(1-x^2)}=\frac{1}{1-x^2}$

Now substitute $\frac{f(y)}{g(y)}$ = v and solve.

Illustration 36: Solve $\frac{dy}{dx}=xy+x^3y^2$

Sol: By rearranging the given equation we will get $\frac{1}{y^2}\frac{dy}{dx}-\frac{1}{y}x=x^3$ and then by putting $\frac{-1}{y}=t$ and using the integration factor we can solve it.

$$\frac{dy}{dx}=xy+x^3y^2 \Rightarrow \frac{dy}{dx}-xy=x^3y^2 \Rightarrow \frac{1}{y^2}\frac{dy}{dx}-\frac{1}{y}x=x^3$$

put $\frac{-1}{y}=t \Rightarrow \frac{dy}{dx}+tx=x^3$

This is a linear differential equation with I.F. = $e^{x^2/2} \Rightarrow t\,e^{x^2/2}=\int e^{x^2/2}x^3dx$

Illustration 37: Find the curve such that the y intercept of the tangent is proportional to the square of ordinate of tangent

Sol: Here X = 0 and Y= y – mx i.e. $x\frac{dy}{dx} - y = -ky^2$. Hence by putting $\frac{-1}{y} = 1$ and applying integration factor we will get the result.

$X = 0 \Rightarrow Y = y - mx \Rightarrow x\frac{dy}{dx} - y = -ky^2$

$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}.\frac{1}{x} = \frac{-k}{x}$

Put $\frac{-1}{y} = t \Rightarrow \frac{dt}{dx} + \frac{t}{x} = \frac{-k}{x}$

$\Rightarrow$ I.f. = x

$\Rightarrow$ Solution is $t.x = -kx + C \Rightarrow \frac{-x}{y} = -kx + C$

7.9 Change of Variable by Suitable Substitution

Following are some examples where we change the variable by substitution.

Illustration 38: Solve $y\sin x\frac{dy}{dx} = \cos x(\sin x - y^2)$

Sol: Here by putting $y^2 = t$, the given equation reduces to $\frac{dt}{dx} + (2\cot x)t = 2\cos x$ and then using the integration factor method we will get result.

$y\sin x\frac{dy}{dx} = \cos x(\sin x - y^2)$

Let $y^2 = t \Rightarrow \frac{1}{2}\sin x\frac{dt}{dx} = \cos x\,(\sin x - t)$

$\Rightarrow \frac{dt}{dx} = 2\cos x - (2\cot x)t \qquad \Rightarrow \frac{dt}{dx} + (2\cot x)t = 2\cos x$

I.F. = $\sin^2 x$

$\Rightarrow$ Solution is $t\sin^2 x = \int 2\cos x.\sin^2 x\,dx$

$y^2\sin^2 x = \frac{2\sin^3 x}{3} + c$

Illustration 39: Solve $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$

Sol: Simply by putting $e^y = t$ and using the integration factor we can solve the above problem.

$\frac{dy}{dx} = e^{x-y}(e^x - e^y) \qquad \Rightarrow e^y\frac{dy}{dx} = (e^x)^2 - e^x e^y$

Put $e^y = t \Rightarrow \frac{dy}{dx} + te^x = (e^x)^2$;

I.F. = $e^{\int e dx} = e^{e^x}$

Solution is $te^{e^x} = \int (e^x)^2.e^{e^x}dx$

(c) Now, sum both the above integrals obtained and quote it to a constant i.e. $\int Mdx + \int Ndy = k$, where k is a constant.

(d) If N has no term which is free from x, the $\int Mdx = c$ (y constant)

Following exact differentials must be remembered:

(i) $xdy + ydx = d(xy)$

(ii) $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$

(iii) $\frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right)$

(iv) $\frac{xdy + ydx}{xy} = d(\log xy)$

(v) $\frac{dx + dy}{x + y} = d\log(x + y)$

(vi) $\frac{xdy - ydx}{xy} = d\left(\ell n\frac{y}{x}\right)$

(vii) $\frac{ydx - xdy}{xy} = d\left(\ell n\frac{x}{y}\right)$

(viii) $\frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-2}\frac{y}{x}\right)$

(ix) $\frac{ydy - xdx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$

(x) $d\left(\frac{e^x}{y}\right) = \frac{ye^x dy - e^x dy}{y^2}$

9. ORTHOGONAL TRAJECTORY

Definition 1: Two families of curves are such that each curve in either family is orthogonal (whenever they intersect) to every curve in the other family. Each family of curves is orthogonal trajectories of the other. In case the two families are identical then we say that the family is self-orthogonal

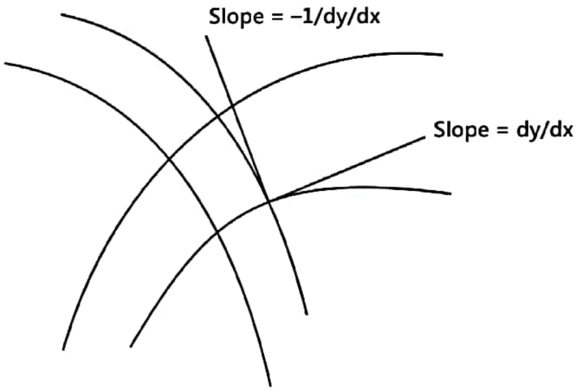

Figure 24.4: Orthogonal trajectories

9.1 How to Find Orthogonal Trajectories

Suppose the first family of curves F(x, y, c) = 0 ... (i)

To find the orthogonal trajectories of this family we proceed as follows. First, differentiate (i) w.r.t. x to find G(x, y, y′, c) = 0 ... (ii)

Now eliminate c between (i) and (ii) to find the differential equation H(x, y, y′) = 0 ... (iii)

If we can write the differential equation in the form

$f(f_1(x, y)\, d(f_1(x, y)) + \phi(f_2(x, y) d(f_2(x,y)) + \ldots\ldots = 0$, then each term can be easily integrated separately. For this the following results must be memorized.

(i) $d(x + y) = dx + dy$

(ii) $d(xy) = xdy + ydx$

(iii) $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$

(iv) $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$

(v) $d\left(\frac{x^2}{y}\right) = \frac{2xydx - x^2dy}{y^2}$

(vi) $d\left(\frac{y^2}{x}\right) = \frac{xydx - y^2dx}{x^2}$

(vii) $d\left(\frac{x^2}{y^2}\right) = \frac{2xy^2dx - 2x^2ydy}{y^4}$

(viii) $d\left(\frac{y^2}{x}\right) = \frac{xydx - 2xy^2dx}{x^4}$

(ix) $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{ydx - xdy}{x^2 + y^2}$

(x) $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$

(xi) $d[\log(xy)] = \frac{xdy + ydx}{xy}$

(xii) $d\left(\log\left(\frac{x}{y}\right)\right) = \frac{ydx - xdy}{xy}$,

(xiii) $d\left[\frac{1}{2}\log\left(x^2 + y^2\right)\right] = \frac{xdx + ydy}{x^2 + y^2}$

(xiv) $d\left[\log\left(\frac{y}{x}\right)\right] = \frac{xdy - ydx}{xy}$

(xv) $d\left(-\frac{1}{xy}\right) = \frac{xdy + ydx}{x^2y^2}$

(xvi) $d\left(\frac{e^x}{y}\right) = \frac{ye^xdx - e^xdy}{y^2}$

(xvii) $d\left(\frac{e^y}{x}\right) = \frac{xe^ydy - e^ydx}{x^2}$

(xviii) $d(x^m y^n) = x^{m-1}y^{n-1}(mydx + nxdy)$

(xix) $d\frac{dt}{dx} + \frac{t}{x}$

(xx) $d\left(\frac{1}{2}\log\frac{x+y}{x-y}\right) = \frac{xdy - ydx}{x^2 - y^2}$

(xxi) $\frac{d\left[f(x,y)\right]^{1-n}}{1-n} = \frac{f'(x,y)}{\left(f(x,y)\right)^n}$

(xxii) $d\left(\frac{1}{y} - \frac{1}{x}\right) = d\left(\frac{1}{y}\right) - d\left(\frac{1}{x}\right) = \frac{dx}{x^2} - \frac{dy}{y^2}$

The differential equation for the other family is obtained by replacing y′ with –1/y′. Hence, the differential equation the orthogonal trajectories is $H(x, y, -1/y') = 0$ ... (iv)

General solution of (iv) gives the required orthogonal trajectories.

Illustration 40: Find the orthogonal trajectories of a family of straight lines through the origin.

Sol: Here as we know, a family of straight lines through the origin is given by $y = mx$.

Hence by differentiating it with respect to x and eliminating m we will get an ODE of this family and by putting –1/y′ in place of y′ we will get an ODE for the orthogonal family.

The ODE for this family is $xy' - y = 0$

The ODE for the orthogonal family is $x + yy' = 0$

Integrating we find $x^2 + y^2 = c$, which are family of circles with center at the origin.

10. CLAIRAUT'S EQUATION

The differential equation

$y = mx + f(m)$, ... (i)

where $m = \frac{dy}{dx}$ is known as Clairaut's equation.

To solve (i), differentiate it w.r.t. x, which gives

$$\frac{dy}{dx} = m + x\frac{dm}{dx} + \frac{df(m)}{dx}$$

$$\Rightarrow \quad x\frac{dm}{dx} + f'(x)\frac{dm}{dx} = 0$$

either $\frac{dm}{dx} = 0 \Rightarrow m = c$ ... (ii)

or $\quad x + f'(x) = 0$ ... (iii)